On non-arithmetic discontinuous groups (Construction of Automorphic Forms and Its Applications)

数理解析研究所講究録 1398 巻 2004 年 241-250

On non-arithmetic discontinuous groups

佐武 一郎 (Ichiro Satake)

In this talk, we will give a survey on arithmetic and non-arithmetic lattices in a semisimple algebraic group. After giving some basic results on the subject, we'll forcus our attention to more recent results, mainly due to Mostow and Deligne, on non-arithmetic lattices in the (projective) unitary group $PU(n,1)$ $(n \geq 2)$. (For more details on these topics as well as the closely related rigidities of lattices, see [S 04]).

1. To begin with, we first fix our settings, giving basic definitions and notations. Let X denote a symmetric Riemannian space of non-compact type (with no flat or compact factors) and let $G = I(X)^o$ be the identity connected component of the isometry group of X. Then, as is well known, G is a connected semisimple Lie group of non-compact type, which is of adjoint type, i.e., with the center reduced to the identity 1. This implies that, denoting by $\boldsymbol{g}$ the Lie algebra of G, one has $G = (\mathrm{Aut}\ \boldsymbol{g})^o$ (^{o} denoting always the identity connected component). The group G acts transitively on X and for any $x_0 \in X$ the stabilizer $K = G_{x_0}$ is a maximal compact subgroup; thus one has $X \cong G/K$. In this manner, G and X determine one another uniquely (up to isomorphisms).

More generally, let G' denote a connected semisimple linear Lie group, which becomes automatically "real algebraic" in the sense that there exists a linear algebraic group $\mathcal{G}$ defined over $\mathbf{R}$ (uniquely determined up to $\mathbf{R}$-isomorphisms) such that $G' = \mathcal{G}(\mathbf{R})^o$. As typical examples, one has $G' = SL(n,\mathbf{R})$, $SO(p,q)^o$, etc. Let K' be a maximal compact subgroup of G', and K'_0 the maximal compact normal subgroup of G'. Then one has

$$G' \supset K' \supset K'_0 \supset (\text{center of } G').$$

Therefore, setting

$$G = G'/K'_0, \quad K = K'/K'_0, \quad X = G/K = G'/K',$$

one obtains a pair (G, X) as described in the beginning; in particular, one has $G = G'$ if K'_0 reduces to the identity group $\{1\}$. We keep these notations throughout the paper.

When $G' = \mathcal{G}(\mathbf{R})^o$, the common dimension r of the maximal $\mathbf{R}$-split tori in $\mathcal{G}$ is called the $\mathbf{R}$-rank of G' and written as $r = \mathbf{R}\text{-rank } G'$. It is well known that, if $\boldsymbol{g}' = \boldsymbol{k}' + \boldsymbol{p}'$ is a Cartan decomposition of $\boldsymbol{g}' =$ Lie G', then

r coincides with the maximal dimension of the (abelian) subalgebras of $\mathfrak{g}'$ contained in $\mathfrak{p}'$. Thus one has $\mathbf{R}$-rank $G' =$ $\mathbf{R}$-rank G.

When the algebraic group $\mathcal{G}$ is defined over $\mathbf{Q}$, G' is said to have a $\mathbf{Q}$-structure and the $\mathbf{Q}$-rank of G' (with this $\mathbf{Q}$-structure) is the common dimension r_0 of the maximal $\mathbf{Q}$-split tori in $\mathcal{G}$. G' is called $\mathbf{Q}$-*anisotropic* when $r_0 = 0$.

2. A subgroup Γ of G' is called a *lattice* in G' if Γ is discrete and the covolume $\mathrm{vol}(\Gamma\backslash G')$ (with respect to the Haar measure of G') is finite. A lattice Γ is called *uniform* if, in particular, the quotient space $\Gamma\backslash G'$ is compact.

Two subgroups Γ and Γ' of G' are said to be *commensurable* if the indices $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are both finite, and one then writes $\Gamma \sim \Gamma'$. As is easily seen, this is an equivalence relation.

A lattice Γ in G is said to be *reducible* if there exists a non-trivial direct decomposition $G = G_1 \times G_2$ such that $\Gamma \sim (\Gamma \cap G_1) \times (\Gamma \cap G_2)$; otherwise, Γ is called *irreducible*. Every lattice in G is commensurable to the direct product of irreducible ones in the direct factors of G.

When $G' = \mathcal{G}(\mathbf{R})^o$ is given a $\mathbf{Q}$-structure, a subgroup Γ of G' commensurable with $\mathcal{G}(\mathbf{Z})$ is called *arithmetic*; the projection of an arithmetic subgroup of G' in $G = G'/K_0'$ is called *arithmetic in a wider sense*. It is clear that arithmetic subgroups (in a wider sense) are discrete.

The following theorem is fundamental.

Theorem 1 (**Borel–Harish-Chandra [BHC 62], Mostow–Tamagawa [MT 62]**) *If Γ is an arithmetic subgroup of G in a wider sense, then Γ is a lattice in G. Moreover, Γ is uniform (i.e., cocompact in G) if and only if G' is $\mathbf{Q}$-anisotropic (i.e.,* $\mathbf{Q}$-rank $G' = 0$).

Note that, when Γ in G is arithmetic only in a wider sense, the $\mathbf{Q}$-rank of G' being $= 0$, Γ is uniform. In the early 1960s it was conjectured by Selberg and others that the converse of Theorem 1 would also be true, if the $\mathbf{R}$-rank of G is high. Actually, we now have

Theorem 2 (**Margulis, 1973, [Ma 91]**) *Suppose that the $\mathbf{R}$-rank of G is ≥ 2. Then any irreducible lattice Γ in G is arithemetic in a wider sense (for a certain choice of G' with a $\mathbf{Q}$-structure).*

3. Thanks to the above result of Margulis, in order to study the arithmeticity of a lattice Γ, we may restrict ourselves to the case $\mathbf{R}$-rank $G = 1$, which naturally implies that G is $\mathbf{R}$-simple. According to the classification of $\mathbf{R}$-simple Lie groups (due to E. Cartan), we have only the following possibilities for (G, X) :

$$G = PU(D; n, 1)^o = U(D; n, 1)^o/(\text{center}), \quad n \geq 2, \; (n = 2 \text{ for } D = \mathbf{O}),$$

$$X = \mathrm{H}^n_D \quad (\text{the hyperbolic } n\text{-space over } D),$$

D denoting a division composition algebra over $\mathbf{R}$, i.e.,

$$D = \mathbf{R}, \; \mathbf{C}, \; \mathbf{H} \text{ (Hamilton's quaternions)}, \; \mathbf{O} \text{ (Cayley's octonions)},$$

and $U(D; n, 1)$ denoting the unitary group of the standard D-hermitian form of signature $(n, 1)$. In the case $D = \mathbf{O}$, which is non-associative, the projective unitary group is defined to be the automorphism group of the (split) exceptional Jordan algebra $\mathrm{Her}_3(\mathbf{O}; 2, 1)$; hence G is of type $\mathrm{F}_{4,1}$.

For $D = \mathbf{R}$, one has $G = SO(n, 1)^o$ (Lorentz group) and $X = \mathrm{H}^n_{\mathbf{R}}$ is the "Lobachevsky space", i.e., the Riemannian n-space of constant curvature $\kappa = -1$, which can be realized by the hyperbolic hypersurface in $\mathbf{R}^{n+1}$ (with the Lorentz metric):

$$\{(x_i) \in \mathbf{R}^{n+1} | \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, \; x_{n+1} > 0 \}.$$

In particular, $\mathrm{H}^2_{\mathbf{R}} (= \mathrm{H}^1_{\mathbf{C}})$ can be identified with the upper half-plane in $\mathbf{C}$ and the lattices in $G = SO(2, 1)^o (\cong SL(2, \mathbf{R})/\{\pm 1\})$ are so-called Fuchsian groups. In this case, it is classical that there are continuous families of non-arithmetic lattices.

For $X = \mathrm{H}^n_{\mathbf{R}}$, $n \geq 3$, non-arithmetic lattices, especially reflection groups, have been studied intensively by E. B. Vinberg and his school since 1965 (see e.g., [V 85], [V 90]). More recently, it was shown by Gromov and Piatetski-Shapiro [GPS 88] that for any $n \geq 2$ one can construct infinitely many non-arithmetic (uniform) lattices as the fundamental group of the "hybrid" of two quotient spaces $\Gamma_1 \backslash X$ and $\Gamma_2 \backslash X$ for non-commensurable arithmetic subgroups Γ_1 and Γ_2 of G.

On the other hand, for the case $D = \mathbf{H}$ and $\mathbf{O}$, Corlette [C 92] and Gromov and Schoen [GS 92] have shown that there exist no non-arithmetic lattices in G by a differential geometric method (harmonic maps), extending the idea of Margulis.

4. In the rest of the paper, we concentrate to the case $D = \mathbf{C}$, i.e., the case where $G = PU(n, 1)$ and $X = \mathrm{H}^n_{\mathbf{C}}$, studied mainly by G. D. Mostow since the early 1970s.

The complex hyperbolic space $H^n_{\mathbf{C}}$ can be realized by the unit ball in $\mathbf{C}^n$ as follows. The unitary group $U(n,1)$ acts on $\mathbf{C}^{n+1}$ and hence on the projective space $P^n(\mathbf{C}) = (\mathbf{C}^{n+1} - \{0\})/\mathbf{C}^\times$ in a natural manner. The orbit of $e_{n+1} = (0,...,0,1)$ (mod $\mathbf{C}^\times$) in $P^n(\mathbf{C})$ is

$$\{z = (z_i) \in \mathbf{C}^{n+1} | \sum_{i=1}^{n} |z_i|^2 - |z_{n+1}|^2 < 0 \}/\mathbf{C}^\times,$$

which, in the inhomogeneous coordinates $z'_i = z_i/z_{n+1}$ $(1 \leq i \leq n)$, is expressed by the unit ball

$$\{z' = (z'_i) \in \mathbf{C}^n | \sum_{i=1}^{n} |z'_i|^2 < 1 \}.$$

The stabilizer of e_{n+1} in $U(n,1)$ is $U(n)\times U(1)$. Hence $H^n_{\mathbf{C}} = U(n,1)/U(n)\times U(1)$ is identified with the unit ball in $\mathbf{C}^n$, on which $G = PU(n,1)$ acts as linear fractional transformations.

We denote by $<>$ the standard hermitian inner product of signature $(n,1)$ on $\mathbf{C}^{n+1}$. For $a \in \mathbf{C}^{n+1}$, $< a, a > > 0$ and $\xi \in \mathbf{C}$, $|\xi| = 1$, we define (after Mostow) a "complex reflection" on $\mathbf{C}^{n+1}$ by

$$R'_{a,\xi} : \quad z \mapsto z + (\xi - 1)\frac{< a, z >}{< a, a >}a \quad (z \in \mathbf{C}^{n+1}).$$

Then, for $\xi, \eta \in \mathbf{C}$, $|\xi| = |\eta| = 1$, one has

$$R'_{a,\xi} \circ R'_{a,\eta} = R'_{a,\xi\eta};$$

in particular, if ξ is a root of unity: $\xi^m = 1$, then one has $(R'_{a,\xi})^m = 1$. We denote the image of $R'_{a,\xi}$ in $G = PU(n,1)$ by $R_{a,\xi}$.

In [M 80] Mostow studied the groups

$$\Gamma = < R_{e_i,\zeta_p} \ (i = 1,2,3) >$$

generated by 3 reflections, where $\zeta_p = e^{2\pi i/p}$ with $p = 3$ or 4 or 5 and

$$e_i \in \mathbf{C}^{n+1}, \ < e_i, e_i >= 1, \ < e_1, e_2 >=< e_2, e_3 >=< e_3, e_1 >= -\alpha\varphi,$$

$$\alpha = (2 \sin\frac{\pi}{p})^{-1}, \quad \varphi = e^{\pi i t/3}$$

with $t \in \mathbf{R}$. Mostow gave a criterion for Γ to be a lattice in G, and found 17 cases, showing that 7 among them are non-arithmetic (i.e., not arithmetic in a wider sense). The non-arithmetic cases are given by

$$[p,t] = [3,1/12], \ [3,1/30], \ [3,5/42], \ [4,1/12], \ [4,3/20],$$

$$[5, 1/5], \ [5, 11/30].$$

(It has turned out that actually the Γ corresponding to $[5, 11/30]$ is arithmetic.)

5. Mostow then studied, in collaboration with Deligne, the analytic construction of lattices in $PU(n,1)$. They consider a system of differential equations of Fuchsian type in n variables, studied for $n=2$ by Picard and in general by Lauricella (1893). The solution space of such equations is $\cong \mathbf{C}^{n+1}$, spanned by the period integrals generalizing the classical Euler integral:

$$F_{g,h}(x_1, \ldots, x_n) = \int_g^h \prod_{i=1}^n (u - x_i)^{-\mu_i} \cdot u^{-\mu_{n+1}} (u-1)^{-\mu_{n+2}} \, du,$$

where

$$\mu = (\mu_1, \ldots, \mu_{n+3}) \in \mathbf{C}^{n+3}, \quad \mu_{n+3} = 2 - \sum_{i=1}^{n+2} \mu_i$$

is the parameter, which we will restrict to the so-called "disc (n+3)-tuple" satisfying the condition $0 < \mu_i < 1$ $(1 \leq i \leq n+3)$, and

$$g, h \in M = \{x = (x_1, \ldots, x_n, 0, 1, \infty) | \ x_i \in \mathbf{C} - \{0, 1\}, \ x_i \neq x_j \text{ for } i \neq j\}.$$

Let $\hat{M}$ be the universal covering space of M. Then there exists a natural map from $\hat{M}$ to $\mathrm{P}^n(\mathbf{C})$, the space of non-zero solutions modulo $\mathbf{C}^\times$, which is equivariant with respect to the actions of the fundamental group on $\hat{M}$ and the projective monodromy group, denoted by Γ_μ, on $\mathrm{P}^n(\mathbf{C})$. It is also shown that there exists a hermitian inner product of signature $(n, 1)$ on the solution space such that Γ_μ is in $PU(n,1)$.

In [DM 86] it was shown that the following condition (INT) is sufficient for Γ_μ to be a lattice in $G = PU(n,1)$.

(INT) If $\mu_i + \mu_j < 1$ with $i \neq j$, then one has $(1 - \mu_i - \mu_j)^{-1} \in \mathbf{Z}$.

Actually, for $n = 2$, this condition is equivalent to the one given by Picard in 1885, so that the 27 lattices obtained in this manner are called "Picard lattices". (In counting the lattices Γ_μ we disregard the order of μ_i's because it is not essential.) There are 9 more μ's satisfying the condition (INT) for $3 \leq n \leq 5$, the longest one being $\frac{1}{4}$ $(1,1,1,1,1,1,1,1)$.

In [M 86] Mostow showed that the following weaker condition (ΣINT) is sufficient to yield the same conclusion.

(ΣINT) One can choose a subset S_1 of $\{1, \ldots, n+3\}$ such that $\mu_i = \mu_j$ for $i, j \in S_1$ and that, if $\mu_i + \mu_j < 1$ with $i \neq j$, one has $(1 - \mu_i - \mu_j)^{-1} \in \frac{1}{2}\mathbf{Z}$ when $i, j \in S_1$ and $\in \mathbf{Z}$ otherwise.

In particular, taking S_1 with $|S_1| = 3$, one obtains Γ_μ commensurable to a lattice generated by 3 reflections, including all lattices constructed in [M 80].

In [M88] Mostow showed further that the converse of the above result is also true in the following sense. First, all Γ_μ which is discrete is a lattice in $PU(n,1)$ (Prop. 5.3) and if $n > 3$ the condition (ΣINT) is necessarily satisfied (Th. 4.13). For $n = 2,3$ there are 10 exceptional lattices Γ_μ with μ not satisfying (ΣINT). The list of all 94 μ's satisfying the condition (ΣINT) is given in [M88], in which the longest one is $\frac{1}{6}(1,1,1,1,1,1,1,1,1,1,1,1)$ with $n = 9$.

6. As for the arithmeticity of Γ_μ, the following criterion was first given in [DM 86] under the assumption (INT):

(A) Let d be the least common denominator of the μ_i's. Then, for all $A \in \mathbf{Z}$, $1 < A < d-1$, $(A,d) = 1$, one has

$$\sum_{i=1}^{n+3} < A\mu_i > = 1 \text{ or } n+2,$$

where $< x >= x - [x]$ for $x \in \mathbf{R}$, $[x]$ being the symbol of Gauss.

It was finally established in [M 88] (Prop. 5.4) that, without any additional assumption, the condition (A) is necessary and sufficient for Γ_μ to be an arithmetic lattice in $PU(n,1)$.

Summing up the above results, we obtain the following

Theorem 3 (Mostow, 1988) *The projective monodromy group Γ_μ is a lattice in $PU(n,1)$ if and only if the condition* (ΣINT) *is satisfied, except for the* 10 *exceptional lattices Γ_μ with $n = 2,3$ not satisfying the condition* (ΣINT). *The group Γ_μ is an arithmetic lattice (in a wider sense) if and only if the condition* (A) *is satisfied.*

In the list of the μ's satisfying (ΣINT) in [M 88], those giving non-arithmetic lattices are marked as NA. (However, this list still seems containing some misprints and erroneous markings.) We give below a (corrected) list of non-arithmetic lattices Γ_μ in $PU(n,1)$, in which the numbering of the μ's is the one given in [M 88].

List of non-arithmetic lattices Γ_μ in $PU(n,1)$

$n=3$

39P	$\frac{1}{12}$(3, 3, 3, 3, 5, 7)		

$n=2$

69P	$\frac{1}{12}$(3, 3, 3, 7, 8)	[4, 1/12]	NA1
71P	$\frac{1}{12}$(3, 3, 5, 6, 7)	(not uniform)	NA2
73P	$\frac{1}{12}$(4, 4, 4, 5, 7)	[6, 1/6]	NA3
74P	$\frac{1}{12}$(4, 4, 5, 5, 6)		NA1
78P	$\frac{1}{15}$(4, 6, 6, 6, 8)	[10, 4/15]	NA4
80	$\frac{1}{18}$(2, 7, 7, 7, 13)	[9, 11/18]	NA5
D7	$\frac{1}{18}$(4, 5, 5, 11, 11)		NA5
84	$\frac{1}{18}$(7, 7, 7, 7, 8)		NA5
85P	$\frac{1}{20}$(5, 5, 5, 11, 14)	[4, 3/20]	NA6
86	$\frac{1}{20}$(6, 6, 6, 9, 13)	[5, 1/5]	NA7
87	$\frac{1}{20}$(6, 6, 9, 9, 10)		NA6
D8	$\frac{1}{21}$(4, 8, 10, 10, 10)		NA9
88	$\frac{1}{24}$(4, 4, 4, 17, 19)	[3, 1/12]	NA8
D9	$\frac{1}{24}$(5, 10, 11, 11, 11)		NA8
89P	$\frac{1}{24}$(7, 9, 9, 9, 14)	[8, 7/24]	NA8
91	$\frac{1}{30}$(5, 5, 5, 22, 23)	[3, 1/30]	NA4
D10	$\frac{1}{30}$(7, 13, 13, 13, 14)		NA4
93	$\frac{1}{42}$(7, 7, 7, 29, 34)	[3, 5/42]	NA9
94	$\frac{1}{42}$(13, 15, 15, 15, 26)	[7, 13/42]	NA9

Remark 1. "P" indicates a *Picard lattice*, i.e. a lattice satisfying (INT). "D" indicates an exceptional lattice, i.e. a lattice not satisfying (ΣINT). For $n=2$, there are 54 lattices (41–94) satisfying (ΣINT) (including 27 Picard lattices) and 9 exceptional lattices (D2–D10).

Remark 2. Γ_μ with $\mu = (\mu_1, ..., \mu_5)$, $S_1 = \{\mu_1, \mu_2, \mu_3\}$, $\mu_4 \le \mu_5$ is commensurable with a reflection group with $[p, t]$, where $p = 2(1-2\mu_1)^{-1}$, $t = \mu_5 - \mu_4$.

7. We say that two subgoups Γ and Γ' of G are *conjugate commensurable* if Γ is commensurable with a conjugate of Γ'. This kind of relations between the Γ_μ's was studied in [M 88], [DM 93]. Some of their results are listed below, where we write $\mu \approx \mu'$ if Γ_μ is conjugate commensurable with $\Gamma_{\mu'}$. It turns out that the 19 non-arithmetic lattices Γ_μ for $n = 2$ are divided into 9 conjugate commensurability classes (NA1–NA9).

It is still an open problem to decide whether or not there exist non-arithmetic lattices not conjugate commensurable to any of Γ_μ, especially such lattices for $n \geq 4$. It would also be interesting to study the *arithmetic* properties of the non-arithmetic lattices Γ_μ, e.g., the corresponding automorphic representations.

(A) ([DM 93], §10) For $a, b > 0,\ 1/2 < a + b < 1$, one has

$$(a,\ a,\ b,\ b,\ 2-2a-2b) \approx (1-b,\ 1-a,\ a+b-\frac{1}{2},\ a+b-\frac{1}{2},\ 1-a-b).$$

In particular, for $a = b$,

$$(a,\ a,\ a,\ a,\ 2-4a) \approx (1-a,\ 1-a,\ 2a-\frac{1}{2},\ 2a-\frac{1}{2},\ 1-2a)$$

$$\approx (\frac{3}{2}-2a,\ a,\ a,\ a,\ \frac{1}{2}-a).$$

Example.

$$\frac{1}{18}(7,7,7,7,8) \approx \frac{1}{18}(11,11,5,5,4) \approx \frac{1}{18}(13,7,7,7,2)$$

$$(\textit{i.e.},\quad 84 \approx D7 \approx 80).$$

For $a + b = 3/4$,

$$(a,\ a,\ b,\ b,\ \frac{1}{2}) \approx (1-b,\ 1-a,\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}).$$

Examples.

$$\frac{1}{12}(4,4,5,5,6) \approx \frac{1}{12}(7,8,3,3,3) \quad (\textit{i.e.},\ 74 \approx 69),$$

$$\frac{1}{20}(6,6,9,9,10) \approx \frac{1}{20}(11,14,5,5,5) \quad (\textit{i.e.},\ 87 \approx 85).$$

(B) For $\pi,\ \rho,\ \sigma$ with $1/\pi +\ 1/\rho +\ 1/\sigma =\ 1/2$, set

$$\mu(\pi,\rho,\sigma) = (\frac{1}{2}-\frac{1}{\pi},\ \frac{1}{2}-\frac{1}{\pi},\ \frac{1}{2}-\frac{1}{\pi},\ \frac{1}{2}+\frac{1}{\pi}-\frac{1}{\rho},\ \frac{1}{2}+\frac{1}{\pi}-\frac{1}{\sigma}).$$

Then ([M 88], Th. 5.6) for $1/\rho + 1/\sigma = 1/6$, one has

$$\mu(3,\ \rho,\ \sigma) \approx \mu(\rho,\ 3,\ \sigma) \approx \mu(\sigma,\ 3,\ \rho).$$

Examples.

$$\rho = 10,\ \sigma = 15: \quad \tfrac{1}{30}(5,5,5,22,23) \approx \tfrac{1}{15}(6,6,6,4,8) \approx \tfrac{1}{30}(13,13,13,7,14)$$

$$(i.e.,\ 91 \approx 78 \approx D10),$$

$$\rho = 8,\ \sigma = 24: \quad \tfrac{1}{24}(4,4,4,17,19) \approx \tfrac{1}{24}(9,9,9,7,14) \approx \tfrac{1}{24}(11,11,11,5,10)$$

$$(i.e.,\ 88 \approx 89 \approx D9),$$

$$\rho = 7,\ \sigma = 42: \quad \tfrac{1}{42}(7,7,7,29,34) \approx \tfrac{1}{42}(15,15,15,13,26) \approx \tfrac{1}{21}(10,10,10,4,8)$$

$$(i.e.,\ 93 \approx 94 \approx D8).$$

References

[BHC 62] A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups, Ann. of Math. **75** (1962), 485-535.

[MT 62] G. D. Mostow and T. Tamagawa, On the compactness of arithmetically defined homogeneous spaces, Ann. of Math. **76** (1962), 446-463.

[M 80] G. D. Mostow, On a remarkable class of polyhedra in complex hyperbolic spaces, Pacif. J. of Math. **86** (1980), 171-276.

[V 85] E. B. Vinberg, Hyperbolic reflection groups, Usp. Math. Nauk **40** (1985), 29-66; = Russian Math. Surveys **40** (1985), 31-75.

[DM 86] P. Deligne and G. D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy groups, Publ. Math. IHES **63** (1986), 5-90.

[M 86] G. D. Mostow, Generalized Picard lattices arising from half-integral conditions, ibid. 91-106.

[M 88] G. D. Mostow, On discontinuous action of monodromy groups on the complex n-balls, J. of AMS **1** (1988), 555-586.

[GPS 88] M. Gromov and I. Piatetski-Shapiro, Non arithmetic groups in Lobachevsky spaces, Publ. Math. IHES **66** (1988), 93-103.

[V 90] E. B. Vinberg, Encyclopedia of Math. Sci., Geometry II, Springer-Verlag. 1990.

[Ma 91] G. Margulis, Discrete Subgroups of Semisimple Lie Groups, Springer-Verlag, 1991.

[C 92] K. Corlette, Archimedian superrigidity and hyperbolic geometry, Ann. of Math. **135** (1992), 165-182.

[GS 92] M. Gromov and R. Schoen, Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one, Publ. Math. IHES **76** (1992), 165-246.

[DM 93] P. Deligne and G. D. Mostow, Commensurabilities among lattices in $PU(1,n)$, Ann. of Math. St. **132**, Princeton Univ. Press, 1993.

[S 04] I. Satake, Mostow (in Japanese), Sugaku Seminar 2/2004, 66-73.